Estimating Property Value by Replicating One

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Abstract

This paper develops a Replication method for estimating property values. Under this method, an appraiser's assignment is to estimate an optimal weight vector that can best replicate the property attributes of a subject property using comparable properties. Once the weights are determined, the value of the subject property is simply a linear combination of the values of the comparable properties based on the estimated weight vector. Since there is no need to estimate hedonic prices (or adjustment factors) of property attributes, the Replication method is superior to the traditional Regression method when there are multicollinearity and mis-speci¯cation problems. The Replication method also outperforms the Grid method when the number of comparables used in the valuation is no less than the number of property attributes identi¯ed for the comparable properties.

1 Introduction

The basic idea of the Replication method is similar to that of the Regression method and the Grid method, although the estimation procedure of these three methods di®er signi¯cantly from each other. The Regression method can be described as a shopping-cart approach. Under this method, each attribute of a property has a price tag. The price of a property is determined by the prices and quantities of the attributes of the property. In other words, when shopping for a property, a person selects the attributes she/he wants for the property (the subject property) and put those attributes into a shopping cart. The customer pays a total price (the estimated property value) at the check-out point based on the attributes (and the price of each attribute) the person selected for the property. When implementing this technique, one must estimate the price tag (or ¯, in statistical terms) for each property attribute. However,it is well-known that the estimation of ¯ could be problematic. It might be fair to say that this estimation problem has prevented the Regression technique from being widely used in the industry.

The Grid method can best be described as a cut-and-paste approach. Similar to the Regression method, the Grid method also assumes that each property attribute has a price tag (could be the same ¯ as that derived in a regression equation). However, to implement the Grid method, one must also ¯nd a number of properties that have similar attributes to the subject property. The Grid method assumes that a person bought a comparable property (with a known price) ¯rst. The person will then sell (or buy) the attributes she/he dislikes (or likes) for the property. After this cut-and-paste process, the comparable property should be similar to the subject property in terms of property attributes. The price of the subject property (the property with the attribute the person wants) should be the price of the comparable property adjusted for the attributes deleted (or added). Since there are several comparable properties, this process should be repeated for each of the comparable properties. At the end, a weighted average of the adjusted comparable prices represents the price that one is willing to pay for the subject property.

As one can see quite clearly, the Grid method requires an appraiser to estimate both the weights for the comparables as well as the price of each property attribute (could be the ¯ obtained from the Regression method). However, when compared with the regression method, the bene¯t of this approach is that an appraiser is able to select more carefully the comparables used for the evaluation of the subject properties. Because of the ability to screen the comparable properties in great details, it is not surprising if the Grid method could perform better than the Regression method. This is especially true if the properties used in estimating β are not really comparable to each other (and, therefore, the estimation equation for β suffers from misspecification problems).

Note that the central theme of both the Regression and the Grid method is to duplicate the subject property by adjusting the attributes of properties. By so doing, both methods require the estimation of the price of each attribute. Our proposed Duplication method takes the same philosophy of duplicating a subject property, but without the need to adjust for the prices of attributes. Specifically, the Replication method replicates the property attributes of the subject property using comparable properties by finding an optimal weight for each of the comparable property. The estimated price of the subject property is then calculated based on the weights and prices of the comparable properties. Clearly, the Replication technique retains the benefits of the Regression and Grid methods, while avoiding the problems associated with the estimation of hedonic prices. (We know that estimations of β could be problematic.)

Section 2 introduces the Replication method while section 3 details the estimation issue. Section 4 and 5 will compare the statistical properties of the Replication method with that of the regression method and Grid method, respectively. The implications of the Replication method are provided in section 6. Section 7 contains our conclusions.

2 The Replication Method

Our proposed Replication method can be accomplished in two simple steps. The first step is to estimate a 1xn optimal vector ω_{rp} such that

$$\omega_{rp} X_C = X_S. \tag{1}$$

X_C is a nxk matrix representing the property attributes of the comparable properties used in the valuation assignment, where n is the number of comparable properties and k is the number of property attributes. X_S is a 1xk vector representing the k property attributes of the subject property. The vector ω_{rp} is a 1xn vector representing the optimal vector of the comparable properties to be estimated by an appraiser. Simply put, at this stage the task of an appraiser is to estimate the optimal vector ω_{rp} that can best duplicate the property attributes of the subject property X_S, using information (on property attributes) from comparable properties X_C.

Once the optimal weight vector ω_{rp} is estimated, the second step of the Replication method is to estimate the value of the subject property using the optimal weights and the prices of comparable properties, or

$$\hat{V}_{S(rp)} = \omega_{rp} V_C, \tag{2}$$

where V_C is a nx1 vector representing the sale prices of the n comparable properties and ω_{rp} is the optimal weight derived from equation (1). We define $\hat{V}_{S(rp)}$ as the estimated value of the subject property using Replication method.

As one can see clearly, the implementation of the Replication method involves only one estimation: the weights for the comparable properties, or ω_{rp}. Intuitively, we know the Replication method should be better than the Grid method since the use of the Grid method involves one more estimation: the adjustment factors β. Although both the Regression method (involves the estimation of β) and the Replication method (involves the estimation of ω_{rp}) require one estimation, the use of the Replication should be preferred since the weight is an exact solution while the estimation of β in the Regression method is based on a best-fit criterion (and is subjected to estimation errors). Additional errors can be introduced through the estimation of the β.

With these in mind, we are now in a position to discuss the implementation issues of the Replication method. This will be done in the next section.

3 Implementation Issues

Equation (1) is basically a system that solves simultaneous k linear equations of ω with n unknown variables. When solving this system of equations, we know that $k = n$ (number of property attributes equals number of comparables) is the required condition for ω_{rp} to be a unique solution of equation (1). If we have more property attributes than comparable properties (or when $k < n$), there is no guarantee that we can find a weight vector that satisfies equation (1). If we have more comparable properties than property attributes (or $k > n$), there will be an infinite number of weight vectors that satisfies equation (1).[1] In the first case, an appraiser just selects the one and only answer as the optimal weight. In the latter two cases ($k < n$ and $k > n$), an appraiser will have to select an optimal weight vector ω_{rp} from alternatives. We will start with the most likely scenario where $k > n$.

3.1 Standard case: when $k < n$

From equations (1) and (2), we know that we can estimate the price of the subject property using $\widehat{V}_{S(rp)} = \omega V_C$ for every ω that satisfy equation (1). In addition, when $k < n$, there are an infinite number of solutions ω that satisfies equation (1). Since property values derived from all weight vectors satisfy equation (1), an appraiser has to impose additional criteria to select the optimal one among alternatives. We propose to use unbiasedness and minimum variance as the criteria.

To implement these criteria, we first make a classic assumption that the true price of any property (subject property or comparable properties) is a linear function of its property attributes, so that

$$V = X\beta + \epsilon, \tag{3}$$

which implies

$$V_C = X_C\beta + \epsilon_C \tag{4}$$

and

$$V_S = X_s\beta + \epsilon_S. \tag{5}$$

X is a $(n+1) \times k$ matrix representing the k property attributes of the subject property and the n comparable properties, β is a kx1 vector of unknown parameters, and ϵ is a 1xk vector of error terms. We also make a standard assumption that the mean of ϵ is a zero vector.

[1] Since every set of k comparables can produce an exact solution, there will be $n!/[(n-k)! \times k!]$ weight vectors that can satisfy equation (1) if there are n comparables. (For example, if we have 5 equations and 4 unknowns, we will have 5 possible solutions when solving this system of equations.) Since linear combinations of those $n!/[(n-k)! \times k!]$ weight vectors can also satisfy equation (1), the solutions are infinite.

Since the true property value is generated via equation (3), given any weight vector w (derived from equation (1), the prediction error should be the difference between the true value V_S (which is unknown) and the estimated value $\hat{V}_{S(rp)} = \omega V_C$ (derived from equation (2)), or

$$\varepsilon_{rp} = V_S - \hat{V}_{S(rp)} = V_S - \omega V_C = X_S\beta + \varepsilon_S - [\omega(X_C\beta + \varepsilon_C)] = \varepsilon_S - \omega\varepsilon_C. \tag{6}$$

It is important to note that in equation (6) the prediction error ε_{RP} is independent of unknown parameters β and relies only on the estimated weight vector ω. That is, there is no error caused by β in the prediction error using the Replication method. Five this, the problems (such as multicollinearity and mis-specifications) occurring in the estimation of β in the Regression method can be circumvented in the Replication method.

Given the zero mean assumption on error terms, the expectation of prediction error ε_{RP} is given by

$$E(\varepsilon_{rp}) = E(\varepsilon_S - \omega\varepsilon_C) = 0, \tag{7}$$

where $E(.)$ is an expectation operator. Equations (7)) show that all Replication estimators have an expected zero error term and are unbiased estimators of the true property value. We can summarize those information into the following proposition.

Proposition 1 Under the assumption that the true property value is a linear function of its property attributes with a zero mean error term (as specified in equation (3)), all Replication estimators satisfying equations (1) and (2) are unbiased.

Since all Replication estimators using weight vectors satisfying equation (1) are unbiased, we will select the optimal weight vector using the minimum variance criterion. To do this, we minimize the expected square prediction error, or

$$\text{Min } E(\varepsilon_S - \omega\varepsilon_C)^2 \tag{8}$$

subject to

$$\omega X_C = X_S.$$

The optimal solution to equation (8), given the zero mean assumption of error terms, is

$$\omega_{rp} = [X_S + \Sigma_{CS}\Sigma^{-1}X_C](X_C^T\Sigma^{-1}X_C)^{-1}X_C^T\Sigma^{-1} - \Sigma_{CS}\Sigma^{-1}. \tag{9}$$

The value of the subject property is

$$\hat{V}_{S(rp)} = \omega_{rp}V_C = [(X_S + \Sigma_{CS}\Sigma^{-1}X_C)(X_C^T\Sigma^{-1}X_C)^{-1}X_C^T\Sigma^{-1} - \Sigma_{CS}\Sigma^{-1}]V_C, \tag{10}$$

where Σ is nxn covariance matrix of ϵ_C, and Σ_{CS} is a 1xn covariance vector between ϵ_C and ϵ_S.

Although equation (10) provides a general estimation of the property value, the covariance terms in equation (10) might be difficult to estimate in reality. To simplify the task (and to make the comparison with the Regression and Grid methods easier), we further assume that the distributions of error terms ϵ_C and ϵ_S are iid with zero mean and standard deviation σ. Under this classic assumption, equation (8) can be simplified as

$$\text{Min } E(\omega\epsilon_C)^2 + \sigma^2 = \text{Min } [\omega\omega^T]\sigma^2 + \sigma^2 \tag{11}$$

subject to

$$\omega X_C = X_S,$$

where σ^2 is the variance of error terms ϵ_C and ϵ_S. The optimal solution of equation (11) is

$$\omega_{(rp,\ iid)} = X_S(X_C^T X_C)^{-1}X_C^T. \tag{12}$$

The price of the subject property is

$$\hat{V}_{S(rp,\ iid)} = \omega_{(rp,\ iid)} V_C = X_S (X_C^T X_C)^{-1} X_C^T V_C. \tag{13}$$

It is clear that when ε_C are iid and un-correlated to ε_S, equation (10) collapses to equation (13). We define σ^2_{rp} as the variance of the prediction errors by the Replication method under the iid assumption, which can be derived by combining equation (11) with equation (13), or

$$\begin{aligned}
\sigma^2_{(rp,iid)} &= [\omega_{(rp,iid)} \omega^T_{(rp,iid)}]\sigma^2 + \sigma^2 \\
&= \sigma^2 + \sigma^2 X_S (X_C^T X_C)^{-1} X_C^T X_C (X_C^T X_C)^{-1} X_S^T \\
&= \sigma^2 + \sigma^2 X_S (X_C^T X_C)^{-1} X_S^T.
\end{aligned} \tag{14}$$

Equation (14) is important as we will compare it with the prediction error derived under the Grid method at a later section. The findings of this sub-section can be summarized in the following proposition.

Proposition 2 Under the assumption that the true property values are a linear function of their property attributes and have error terms distributed iid with a zero mean, when the number of comparable properties is more than the number of property attributes used in the evaluation the optimal weight set of a Replication estimator is

$$\omega_{rp} = X_S (X_C^T X_C)^{-1} X_C^T,$$

the value of the subject property is

$$\hat{V}_{S(rp)} = X_S (X_C^T X_C)^{-1} X_C^T V_C,$$

and the variance of the prediction error of a Replication estimator is

$$\sigma^2_{rp} = \sigma^2 + \sigma^2 X_S (X_C^T X_C)^{-1} X_S^T.$$

3.2 Special cases: when $k = n$ and $k > n$

When the number of properties equals the number of property attributes $n = k$, there exists only a unique solution for equation (1). Under this circumstance, equation (1) can be estimated without uncertainty and we can rewrite equation (1) as

$$V_C = X_C \beta. \tag{15}$$

Similarly, true value of the subject property V_S can be re-defined as

$$V_S = X_S \beta. \tag{16}$$

When ω_{rp} is the unique solution that satisfies $\omega_{rp} X_C = X_S$, then

$$\hat{V}_S = \omega_{rp} V_C = \omega_{rp} X_C \beta = X_S \beta = V_S. \tag{17}$$

This simple exercise indicates that, as long as property price is a linear function of property attributes and the number of comparables is equal to the number of property attributes, the true property value should be the same as that predicted by the proposed Replication method. Clearly, this is the most desirable condition that anyone can hopeful.

When there are more property attributes (equations) than the number of comparable properties (variables), or when $k > n$, there is no solution for equation (1) in general. In other words, there is no guarantee that one is able to replicate a subject property using the limited number of comparable properties.[2] However, when there is no exact solution for equation (1), appraisers can still find an optimal solution of weight vector for estimating property values.

The ideas is to create a property (using property attributes of comparable properties) that is the most similar to the property attributes of the subject property. In other words, while we cannot completely replicate the property attributes of the subject property, we can mimic the property attributes of the subjective property in terms of minimum sum of square error, or

$$\begin{aligned} & \text{Min } E(\epsilon_S - \omega_{rp}\epsilon_C)^2 && (18) \\ = \; & (X_S - \omega_{rp} X_C)(X_S - \omega_{rp} X_C)^T, && (19) \end{aligned}$$

[2] While it is possible that we can duplicate the subject property when $k > n$, in this case the solution will be the same as that discuss in the scenario $k < n$ or $k = n$. We, therefore, ignore this possibility.

where X_C^T is a kxn transpose matrix of X_C. The optimal solution ω^*_{rp} that minimizes the prediction error defined by equation (18) is

$$\omega^*_{rp} = X_S X_C^T (X_C X_C^T)^{-1}. \quad (20)$$

The value of the subject property, under this scenario, can be estimated as

$$\hat{V}_{S(rp)} = \omega^*_{rp} V_C = X_S X_C^T (X_C X_C^T)^{-1} V_C. \quad (21)$$

Although ω^*_{rp} in equation (20) is not an exact solution for equation (1), $\omega^*_{rp} X_C$ represents best mimic attributes of the subject property in term of the least square error. In other words, while we do not have an exact answer because of the constraint, the assignment is to use the best answer one can find as the solution of the problem.[3]

3.3 Practical issues

While all the three scenarios ($k > n$, $k = n$, and $k < n$) are discussed in this section, we believe that, in practice, the scenario $k < n$ might be the one that should receive the most attention. Clearly, when the number of the comparables is no more than the number of property attributes, we will not be able to estimate hedonic price vector $\hat{\beta}$ using a regression equation. Since $\hat{\beta}$ cannot be estimated, it is not feasible to use both the regression method and the Grid method.[4] While we might be able to use the Replication method, we will not able to ascertain how the estimator performs when compared with the regression and the Grid method.

Given this, In the following sections we will only concentrate our discussion on the scenario $k < n$. We will first show that equation (13) provides exactly the same estimator as that from the Regression method. Since the Regression method uses the same zero mean and iid assumptions on error

[3] When applying the Arbitrage Pricing Model using real world data, Huberman, Kandel and Stambaugh (1987) demonstrate that it is feasible to use mimicking portfolios to substitute the k factors in an exact Arbitrage Pricing Model. In this regard, our approach is similar to their approach.

[4] While it might be feasible to use other subjective methods to derive the adjustment factors for the Grid method, we will not be able to make meaningful comparisons with the Regression or the Replication methods under this circumstance since those subjective methods might not have a closed-form equation.

terms, our Replication approach can provide at least as good an estimator as the Regression method. However, since the Replication does not require the estimation of β, we will show that our Replication approach is better than the Regression method if equation (3) suffers from multicollinearity and misspecification problems. We will also show that our Replicated method is superior to the Grid method, as long as the number of comparable properties is more than the number of property attributes used in the valuation assignment.

4 Comparing with the Regression Method

We will compare the Replication estimator with the estimators derived from the traditional Regression and Grid methods under the same set of assumptions. In other words, we start with the same assumption of a linear pricing model, or

$$V_C = X_C\beta + \epsilon_C,$$

where β is an unknown kx1 vector that contains the hedonic coefficients of property attributes (which can also be referred to as the adjustment factors of property attributes). Since β is an unknown vector, an appraiser has to estimate it first. It is well-known that, under the Ordinary Least Square (OLS) assumption,

$$\hat{\beta} = (X_C^T X_C)^{-1} X_C^T V_C, \tag{22}$$

where X_C^T is the transpose of X_C and $\hat{\beta}$ is an unbiased estimator of the true β. The value of the subject property can be estimated using the estimated hedonic coefficients $\hat{\beta}$ and property attributes of the subject property, or

$$\hat{V}_{S(ols)} = X_S\hat{\beta} = X_S(X_C^T X_C)^{-1} X_C^T V_C, \tag{23}$$

where $\hat{V}_{S(ols)}$ is the estimated value of the subject property using the Regression method. The prediction error of a Regression estimator is

$$\varepsilon_{ols} = V_S - \hat{V}_{S(ols)} = X_S\beta + \varepsilon_S - X_S\hat{\beta} = \varepsilon_S - X_S(\hat{\beta} - \beta). \qquad (24)$$

It is important to note that the estimator provided by equation (23) is exactly the same as that in equation (13). This means that, when the linear pricing model specified in equation (3) holds and error terms are distributed iid with a zero mean, the Replication method provides exactly the same estimator as that provided by the Regression method.

However, it is well known that equation (3) might not hold in applications. We might face multicollinearity problem. We might also omit relevant variables or include irrelevant variables in actual applications. How well does the Replication method perform when compared with the regression method under those circumstances?

4.1 Multicollinearity

Furthermore, the multicollinearity in the Regression model causes the estimated adjustment factor's variance (because the variation of inverse matrix $(X_C^T X_C)^{-1}$of explanatory variable (for detailed, see page 239 in Johnston) and could increase the variance of prediction error. However, the solutions of solving a linear systematic equations (2) will not be affected as much as significant as in the Regression model because the stability of the linear systematic equations. This implies the multicollinearity may cause the problem in using the Regression model but not in the replication because how to estimate the adjustment factor is totally irrelevant in the prediction of the subjective property value in the replication method.

4.2 Omitted variables

Again, using the same assumption as in the Replication method that the $V_S = X_s\beta + \varepsilon_S$ (equation (5), the prediction error of the Regression method is

It is important to know that the prediction error of the Regression depends results from the error in the estimated hedonic coefficients $(\hat{\beta} - \beta)$. The expectation of the prediction error is given by

$$E[\varepsilon_S - X_S(\hat{\beta} - \beta)] = -X_S[E(\hat{\beta}) - \beta] \neq 0$$

The non-zero expectation of prediction error in equation (24) implies, the biased of $\hat{\beta}$ would produce the biased prediction of the subjective property value.

Equation (24) show that using regression could produce a biased prediction if the adjustment factor is biased estimated, however, as shown in equation (4), the replicated method always produce the unbiased prediction on the subjective property value once the error terms' expectation is zeros, which is a common assumption in most of uncertainty model.

4.3 Irrelevant variables

higher variance in β:

Proposition 3 Under the assumption that the true property values are a linear function of their property attributes and have error terms distributed iid with a zero mean, a Replication estimator perform at least the same as a Regression estimator. When a regression equation suffers from multicollinearity and/or mis-specification problems, a Replication estimator performs better than a Regression estimator.

5 Comparing with the Grid Method

The Grid method also uses a weighted average of the observed comparable property prices (after adjusted for differences in property attributes) to estimate the value of the subject property. Define $V_{S(grid)}$ as the estimated value of the subject property using the Grid method, the estimation equation of the Grid method can be written as

$$\hat{V}_{S(grid)} = \omega_{grid}(V_C + (X_S - X_C)\hat{\beta}) \tag{25}$$

and

$$\omega_{grid}e = 1,$$

where e is nx1 vector, ω_{grid} is the weight vector derived under the Grid method, and $\hat{\beta}$ is the adjustment-factor vector (the prices for property attributes). When implementing the Grid method, an Appraiser's assignment is to select the number of comparables n and to estimate both ω_{grid} and $\hat{\beta}$. While there is no consensus on how the weight vector ω_{grid} should be estimated,[5] it is often agreed that adjustment-price vector $\hat{\beta}$ can be the same as that derived from the Regression method.

Under the same assumptions that the true property value is a linear function of its property attributes with a zero mean error term (see equation (3)), the prediction error of a Grid estimator ϵ_{grid} can be specified as

$$\begin{aligned}
\epsilon_{grid} &= V_S - \hat{V}_{S(grid)} = V_S - \omega_{grid}(V_C + (X_S - X_C)\hat{\beta}) && (26)\\
&= V_S - [\omega_{grid}(V_C - X_C(X_C^T X_C)^{-1}X_C^T V_C)] - X_S(X_C^T X_C)^{-1}X_C^T V_C && (27)\\
&= X_S\beta + \epsilon_S - \omega_{grid}[I_n - X_C(X_C^T X_C)^{-1}X_C^T](X_C\beta + \epsilon_C) && (28)\\
&\quad - X_S(X_C^T X_C)^{-1}X_C^T(X_C\beta + \epsilon_C) \\
&= \epsilon_S - \omega_{grid}[I_n - X_C(X_C^T X_C)^{-1}X_C^T]\epsilon_C - X_S(X_C^T X_C)^{-1}X_C^T\epsilon_C. && (29)
\end{aligned}$$

Under the same assumption of uncorrected error terms ϵ_S and ϵ_C, the variance of the prediction errors of a Grid estimator can be calculated from equation (29) as

$$\begin{aligned}
\sigma^2_{grid} &= \sigma^2 + \sigma^2\{\omega_{grid}[I_n - X_C(X_C^T X_C)^{-1}X_C^T][I_n - X_C(X_C^T X_C)^{-1}X_C^T]\omega_{grid}^T \\
&\quad + X_S(X_C^T X_C)^{-1}X_S^T\} + 2X_S(X_C^T X_C)^{-1}X_C^T E(\epsilon_C\epsilon_C^T)[I_n - X_C(X_C^T X_C)^{-1}X_C^T]\omega_{grid}^T \quad (30)
\end{aligned}$$

Since $I_n - X_C(X_C^T X_C)^{-1}X_C^T$ is an idempotent matrix, equation (30) can be re-written as

$$\begin{aligned}
\sigma^2_{grid} &= \sigma^2 + \sigma^2\{\omega_{grid}[I_n - X_C(X_C^T X_C)^{-1}X_C^T]\omega_{grid}^T + X_S(X_C^T X_C)^{-1}X_S^T\} \\
&\quad + 2\sigma^2 X_S(X_C^T X_C)^{-1}X_C^T[I_n - X_C(X_C^T X_C)^{-1}X_C^T]\omega_{grid}^T. && (31)
\end{aligned}$$

Since we know that $X_C^T[I_n - X_C(X_C^T X_C)^{-1}X_C^T] = X_C^T - X_C^T X_C(X_C^T X_C)^{-1}X_C^T = 0$, equation (31) can be rewritten as

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[5] Other than the constraint that $\omega_{grid}e = 1$, there is no definite rule on how the weight vector ω_{grid} should be selected. Practitioners insert that one should give more weights to the most comparable properties, but have a difficult time defining what are most comparable properties or giving an exact formula for determining the weights. Academicians, on the other hand, derive some exact methods for estimating weights. However, few practitioners use them. (See, for example, Colwell, Canaday and Wu (1983), Vandell (1991), Gau, Lai and Wang (1992), Isakson (1986 and 2002), and Pace (1998) for a detailed discussion on the different methods of estimating ω_{grid}).

$$
\begin{aligned}
\sigma^2_{grid} &= \sigma^2 + \sigma^2\{\omega_{grid}[I_n - X_C(X_C^T X_C)^{-1}X_C^T]\omega_{grid}^T + X_S(X_C^T X_C)^{-1}X_S^T\} \qquad (32)\\
&> \sigma^2 + \sigma^2\{X_S(X_C^T X_C)^{-1}X_S^T\}.
\end{aligned}
$$

Equation (32) is greater than $\sigma^2 + \sigma^2 X_S(X_C^T X_C)^{-1}X_S^T$ because the term $\omega_{grid}(I_n - X_C(X_C^T X_C)^{-1}X_C^T)\omega_{grid}^T$ must be positive if ω_{grid} is not a zero vector. We know that ω_{grid} cannot be a zero vector because the sum of the weights of comparable properties must be equal to one.

Equation (32) provides an important result of our paper. It is important to note that $\sigma^2 + \sigma^2 X_S(X_C^T X_C)^{-1}X_S^T$ is, in fact, the variance of the prediction errors under the Replication method (see equation (14)). Since the variance of the prediction errors of the Grid method is always greater than that of the variance of the prediction errors of the Replication method, the proposed Replication method should be a preferred when compared to the Grid method. This is especially true if we include the unbiasedness criterion into consideration. We know that a Replication estimators is always unbiased. However, from equation (??), it is clear that a Grid estimator is unbiased only if the estimated adjustment factor $\hat{\beta}$ is an unbiased estimator of the true β. This condition might not hold all the time and a Grid estimator could be a biased estimator of the true property value. The following proposition summarizes our finding in this section.

Proposition 4 Under the assumption that the true property values are a linear function of their property attributes and have error terms distributed iid with a zero mean, the variance of the prediction error derived from the Replication method is smaller than that from the Grid method as long as the number of comparables is no less than the number of attributes used in the valuation.

6 Implications of the Results

If the number of comparable is more than the number of attributes and no prior information on the error terms, an appraiser may include more attributes in the model to increase the number of attributes k until $k = n$. For example, appraisers can add the square of attributes or the cross product of attributes, etc. in the model. Including the higher order of attributes and the cross products to the non-linear models and would result a better prediction than linear models. According to Taylor expansion, any function can be approximately by a polynomial function. Hence, if the valuation generating function is a non-linear function of attributes, then using Taylor expansion to include higher degree terms of attribute would improve its accuracy. Since there is no multicollinearity problem in our approach, increasing the number of attributes (e.g., created by higher degree of attributes and the cross product of attributes) could increase the accuracy of the prediction on the subjective property value.

7 Conclusions

8 References

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