The Pricing of Purchase Discount Options in Taiwanese Housing Market

Yao-Min Chiang*
Department of Finance
National Chengchi University, Taiwan
Email: ymchiang@nccu.edu.tw

Chia-Hsin Huang
Department of Finance
National Yunlin University of Science and Technology, Taiwan
Email: sing@yuntech.edu.tw

Jay Sa-Aadu
Department of Finance
University of Iowa
Iowa City, IA 52242
Email: Jsa-aadu@uiowa.edu

Abstract

In this paper, we design a house purchase discount option to promote house sale. We are also able to provide the closed-form pricing formula for these new options. This new designed discount option is especially suitable for using in a house pre-sale system. This option can help to promote house sale and also help to stabilize the developer's stock price. The purchase discount options are function of two dynamic price process of house and stock. We follow Asian option pricing approach to derive the pricing formula. This kind of purchase discount options has potential for wide application.

Keywords: house sale, discount, options, pricing model, closed-form solution.

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* Corresponding author.
 Comments are welcome. We are grateful to Son-Nan Chen for his valuable comments.

I. INTRODUCTION

In July 2001, one Taiwanese developer, SamMon, used a stock and house price exchange program to promote its project. Homebuyers sell stocks in their portfolio, and use 22.50%[1] of the amount to be the house purchase discount. Once homebuyers sold their stocks, they will have enough cash to buy the house and can also get some discount. This approach therefore helped the developer to sell houses. However, this simple form of exchange provides only small effect on home sales. Also, there is no help on pushing up the developer's own stock price. In this paper, we create a more complicate way to further link stock market and housing market. We believe there is no similar discussion in the literature.

In this paper, we design a purchase discount options that can help the developer to sell the house and to pull up its own stock price. Investors buy the developer's stock and enter the option contract by paying premium. The house purchase price, also the exercise price, is determined when the option contract is signed. The maturity of the option is one year. When investors decide to exercise the option, they pay the pre-decided price and get a certain amount of discount. The discount is based on a certain percentage on the stock price at the time when investors decide to exercise their options. Investors are assumed to sell the developer's stock in order to get enough money to buy the product. This option is able to push up the developer's

[1] There is a price limit of 7% in Taiwan's stock market. $(1+7\%)^3-1=22.50\%$.

stock price and to promote the product[2]. This is a new way to promote pre-sale house. Surveying the market, we can only see one similar idea being applied in Taiwan. The purchase discount option we provide in this paper is a new design. We have found no literature dealing with these problems.

In the paper, we also derive the pricing formula for these purchase discount options. A closed-form pricing formula in terms of a multivariate normal distribution is derived under the risk-neutral framework. The purchase discount options are related to two assets, house and stock. A feature of these options is like that of the Asian options. We follow Vorst's (1992) approximation via geometric average to derive the pricing formula.

This paper is organized as follows. The second section briefly describes the house market in Taiwan. Section three describes the purchase discount options design. We provide a closed-form solution for the purchase discount options in section four. The last section is conclusion and suggestion.

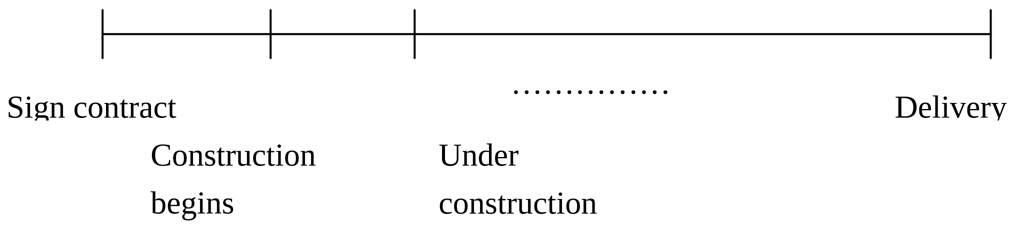
II. HOUSING MARKET IN TAIWAN

Over the past decade, Taiwan's housing market has been suffering a serious recession. It has become harder to sell a house. The length of house selling

[2] This option can be redesigned to replace the manufacture' stock to any stock or an index.

process is increasing. It was 44 days to close a sale in the year of 1995, but it increased to 60 days in 2000. Sale deterioration hurts developers' performance and the firms' stock price. Before 1998, construction stock price index commoved with Taiwan stock price index. However, after 1998, Taiwan stock price index had several booms, and construction stock price index continued to decrease. Most developers' stock prices were only NT$2 or 3, much lower than the face value of NT$10. Developers and investors really hoped there is a way to push the developers' stock prices up.

The pre-sale system began to become popular in 1973. Unlike in Hong Kong, developers in Taiwan are allowed to sell residential buildings before they are started to construct or are under construction. The length for starting selling through project completion is about 24 months. There are four important steps that homebuyers need to install their payments.



Homebuyers pay their down payment before the property is delivered.

Currently down payment is 30% of the house price. When homebuyers sign the contract, they have to pay some premium. The second important step is the starting construction. Homebuyers have to pay some more money. Homebuyers also need to pay installment when the projects are under construction. Once they properties are completed and are ready for delivery, homebuyers have to complete the down payment, and use bank loan to pay 70% of the house price.

This pre-sale system helps homebuyers to finance their purchase. Homebuyers do not need to pay all of the down payment at one time. They can make an installment. Also, homebuyers can ask the developers to make some design change before the building starts constructing.

The pre-sale system provides developers a way to finance the projects. Therefore, a lot of small developers helped to supply residential properties in Taiwan. This system reduces the hurdle of new entry of suppliers and thus enhances competition. Small developers are able to use small capital to compete with bigger, well-financed developers. However, pre-sale housing also associates with complaints from home buyers in which poor quality of construction. And sometimes, developers may stop the projects when they consider there is no profit to make.

Wang, Zhou, Chan, and Chau (2000) consider over-confidence resulted from pre-sale system contributed to the oversupply problem in Hong Kong. We have no evidence to conclude same thing happened in Taiwan. However, oversupply problem really hurts the survival of developers in Taiwan. It is estimated that there is oversupply of more than 1 million house units in the market at the end of 1999. To meet with this challenge, developers have to think of new ways to promote their projects.

In July, 2001, SamMon Developing Company used a new promotion way in their pre-sale system. They provide a way to grand discount to homebuyers. Homebuyers can sell stocks from their investment portfolio and provide the sale receipt to the developer. The developer will grand 22.5% of the amount homebuyers sold the stocks as a discount to the house price. This approach is to encourage potential buyers to cash their stock investment and use the money to purchase the house. Homebuyers may be lack of cash, but they have investment in the stock market. They have cash constraint. The developer gave the homebuyers an incentive to sell stocks and come to buy the house. This promotion approach helped the developer to sell their houses. However, this approach has no linkage between the developer’s stock price and house purchase. We need to find a new way that can help the developer to sell the house and to pull up its own stock price.

III. A HOUSE PURCHASE DISCOUNT OPTIONS

The incentive to design a house purchase discount option is to link the house sale with the developer’s stock. We want potential buyers to buy the deveoper’s stock and get house purchase discount based how much they paid for the stock purchase. Homebuyers show the stock purchase receipt to the developer and get house purchase discount. Homebuyers then can sell the stocks to pay for the house down payment.

Model A: Purchase Discount

Prospective homebuyer invests in the stock of the developer’s company, simultaneously signs a contract to buy a house now and receives 22.5% discount on the amount invested in the stock of the company as discount towards the purchase price of the house.

- Buy stocks now, sign the contract to buy a house, using 22.50% of the amount invested in stock as the discount, the ceiling of the discount is 50% of the house price. i.e. Discount= $\min(S_0((1+7\%)^3 - S_0, \frac{P_0}{2})$
- current stock price: S_0
- net purchase price: $P_0 - \min(S_0((1+7\%)^3 - S_0, \frac{P_0}{2})$
- If homebuyer sells stocks right after the contract has been signed, there is no gain or loss on purchasing stocks. The net cost saving for homebuyers is

$\min(S_0((1+7\%)^3 - S_0, \frac{P_0}{2})$。

Model B: Purchase Discount Options

The potential homebuyer buys the developer' stocks, which entitles the investor to purchase a house from the developer where the option is exercised a year from now. The house price is pre-decided at time the option contract is entered, with the exercise to occur are a later date. Investors buy stock now, and sign an option contract to purchase the house in a year. When buyers decide to exercise the option, they get 22.50% of the stock's market value as the house purchase discount. The ceiling of the discount is 50% of the initial purchase price.

- Current stock price: S_0
- Signing the contract: Housing price is determined P_0
- Option's maturity: 1 year.
- Option premium: C. It is the premium to sign the option contract. When homebuyers decide to purchase the house, the premium can be a part of the house payment. If investors decide not to exercise the options, they lose the money to the developers.
- Time t: When investors decide to exercise the options, they can get 22.50% of the stock market value as the house purchase discount. The ceiling of the discount is 50% of the exercise price.
- Purchase discount: $\min(S_t(1+7\%)^3 - S_t, \frac{P_0}{2})$ 。

- Net house price: $P_0 - \min(S_t(1+7\%)^3 - S_t, \frac{P_0}{2})$

A generalized model

Homebuyers buy the developer's stock, nS_0, and enter the option contract by paying premium. The house purchase price, also the exercise price, $K = P_o$, is determined when the option contract is signed. The maturity of the option is one year. Investors can exercise the option at any time before the maturity. When investors decide to exercise the options, they pay the pre-decided price and get a certain amount of discount, $\min(bnS_t, dP_o)$. The discount is based on a certain percentage, b, on the stock price at the time when investors decide to exercise their options and on a certain percentage, d, of the pre-decided house price. Investors are assumed to sell the developer's stock in order to get enough money to buy the product.

- Investors buy the developer's stocks. n shares; current stock price: S_0; total amount nS_0.
- Signing the option contract: The product's price is determined at P_0. Investors pay premium C to sign the option contract.
- Option's maturity: 1 year. Investors can exercise the option at any time before the maturity.

- At time t, when investors decide to exercise the options,

1. They pay the pre-decided price, P_0, to buy the product.
2. They can get a certain percentage, b, of the stock market value as the house purchase discount. The ceiling of the discount is a certain percentage, d, on of the initial house price.
3. They can sell stocks at the price of S_t, getting nS_t dollars to pay the down payment.

- Purchase discount: $\min(bnS_t, dP_0)$ 。
- Net house price: $P_0 - \min(bnS_t, dP_0)$
- Payoff: $nS_t + \max[P_t - (P_0 - \min(bnS_t, dP_o)),0]$

The portfolio,

strategies	t=0	t=t
1.buy stocks	nS_0	nS_t
2. long a call	c	$\max[P_t - (P_0 - \min(bnS_t, dP_o)),0]$

IV. PURCHASE DISCOUNT OPTION PRICING MODEL

An investor buys a developer's common stocks. The developer grants the investor an option. The option allows the holder to buy the developer's house with an exercise price, $K = P_o$, and with a discount of $\min(bnS_t, dP_0)$, matured in a year.

b is a constant. n is number of shares purchased. d is a constant.

Payoff of the call options:

$$C_T = \max[P_t - (K - \min(bnS_t, dP_0)),0]$$

$$C_T = \begin{cases} P_t - K + dP_0 & \text{if } bnS_t \geq dP_0, P_t \geq K - dP_0 \\ P_t + bnS_t - K & \text{if } bnS_t < dP_0, P_t + bnS_t \geq K \\ 0 & \text{if } bnS_t \geq dP_0, P_t < K - dP_0 \\ 0 & \text{if } bnS_t < dP_0, P_t + bnS_t < K \end{cases}$$

Deriving the Pricing Model

1. We derive the pricing model for the European-style options.

2. For simplicity, we view bnS_T as S_T and dP_o as d

$$C_T = \max[P_T - (K - \min(S_T, d)),0]$$

$$= \max[P_T - K + d - \max(d - S_T, 0),0]$$

$$C_T = \begin{cases} P_T - K + d & \text{if } S_T \geq d, P_T \geq K - d \\ P_T + S_T - K & \text{if } S_T < d, P_T + S_T \geq K \\ 0 & \text{if } S_T \geq d, P_T < K - d \\ 0 & \text{if } S_T < d, P_T + S_T < K \end{cases}$$

We look at the classical setting where besides a riskless bank account with constant interest rate r, our arbitrage-free market model comprises two assets, stock and house, whose prices at time t are denoted by S_t and P_t. We assume that their risk-neutral price dynamics are given by the following stochastic differential equations:

$$\frac{dS_t}{S_t} = rdt + \sigma_s \, dW_{St}$$
$$\frac{dP_t}{P_t} = rdt + \sigma_p \, dW_{Pt}$$

Where volatilities σ_s and σ_p are positive constants and W_S and W_P are two Brownian motions with correlation ρ. The initial conditions will be denoted by S_0 and P_0. The price of the purchase option discount, C at time 0 with date of maturity T is given by the risk-neutral expectation:

$$C = e^{-rT} E^Q[(P_T - K + d) I_{\{S_T \geq d, P_T \geq K-d\}}] + e^{-rT} E^Q[(P_T + S_T - K) I_{\{S_T < d, P_T + S_T \geq K\}}]$$

According to Girsanov Theorem,

$$dW_1^Q = dW_1^R + \sigma_1 dt$$

$$dW_2^Q = dW_2^R + \sigma_1 \rho dt$$

➔when viewing S as the first asset, and P as the second asset, we have

$$S_T = S_0 \exp[(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q] = S_0 \exp[(r + \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^R]$$

$$P_T = P_0 \exp[(r - \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^Q] = P_0 \exp[(r - \sigma_p^2/2)T + \rho \sigma_s \sigma_p T + \sigma_p \Delta W_{pT}^R]$$

➔when viewing P as the first asset, and S as the second asset, we have

$$P_T = P_0 \exp[(r - \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^Q] = P_0 \exp[(r + \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^R]$$

$$S_T = S_0 \exp[(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q] = S_0 \exp[(r - \sigma_s^2/2)T + \rho\sigma_s\sigma_p T + \sigma_s \Delta W_{sT}^R]$$

The first part of the price formula.

1. $e^{-rT} E^Q[(P_T - K + d) I_{\{S_T \ge d, P_T \ge K-d\}}]$

$$e^{-rT} E^Q[(P_T - K + d) I_{\{S_T \ge d, P_T \ge K-d\}}]$$

$$= e^{-rT} E^Q[P_T I_{\{S_T \ge d, P_T \ge K-d\}}] \ \text{-} \ e^{-rT} E^Q[(K-d) I_{\{S_T \ge d, P_T \ge K-d\}}]$$

a. $e^{-rT} E^Q[P_T I_{\{S_T \ge d, P_T \ge K-d\}}]$

$$e^{-rT} E^Q[P_T I_{\{S_T \ge d, P_T \ge K-d\}}] = e^{-rT} P_0 e^{rT} E^Q[e^{-\sigma_p^2/2 + \sigma_p \Delta W_{pT}^Q} I_{\{P_T \ge K-d, S_T \ge d\}}]$$

$$= P_0 P_r^R (P_T \ge K - d, S_T \ge d)$$

$$P_T \ge K - d$$

➔ $P_0 \exp[(r + \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^R] \ge K - d$

➔ $[(r + \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^R] \ge \ln(K - d/P_0)$

➔ $-\dfrac{\Delta W_{pT}^R}{\sqrt{T}} \le \dfrac{\ln(P_0/K-d) + (r + \sigma_p^2/2)T}{\sigma_p \sqrt{T}} = f_1$

$$S_T \ge d$$

➔ $S_0 \exp[(r - \sigma_s^2/2 + \rho\sigma_s\sigma_p)T + \sigma_s \Delta W_{sT}^R] \ge d/S_0$

➔ $(r - \sigma_s^2/2 + \rho\sigma_s\sigma_p)T + \sigma_s \Delta W_{sT}^R] \ge \ln(d/S_0)$

➔ $-\dfrac{\Delta W_{sT}^R}{\sqrt{T}} \le \dfrac{\ln(S_0/d) + (r - \sigma_s^2/2 + \rho\sigma_s\sigma_p)T}{\sigma_s \sqrt{T}} = g_1$

$\therefore e^{-rT} E^Q [P_T I_{\{S_T \ge d, P_T \ge K-d\}}] = P_0 BN(f_1, g_1, \rho)$

b. $e^{-rT} E^Q [(K-d) I_{\{S_T \ge d, P_T \ge K-d\}}]$

$e^{-rT} E^Q [(K-d) I_{\{S_T \ge d, P_T \ge K-d\}}] = e^{-rT} (K-d) Q(S_T \ge d, P_T \ge K-d)$

$P_T \ge K-d$

➔ $P_0 \exp[(r - \sigma_p^2 / 2)T + \sigma_p \Delta W_{pT}^Q] \ge K-d$

➔ $[(r - \sigma_p^2 / 2)T + \sigma_p \Delta W_{pT}^Q] \ge \ln(K - d / P_0)$

➔ $-\dfrac{\Delta W_{pT}^Q}{\sqrt{T}} \le \dfrac{\ln(\dfrac{P_0}{K-d}) + (r - \sigma_p^2 / 2)T}{\sigma_p \sqrt{T}} = f_2$

$S_T \ge d$

➔ $S_0 \exp[(r - \sigma_s^2 / 2)T + \sigma_s \Delta W_{sT}^Q] \ge d / S_0$

➔ $(r - \sigma_s^2 / 2)T + \sigma_s \Delta W_{sT}^Q] \ge \ln(d / S_0)$

➔ $-\dfrac{\Delta W_{sT}^Q}{\sqrt{T}} \le \dfrac{\ln(S_0 / d) + (r - \sigma_s^2 / 2)T}{\sigma_s \sqrt{T}} = g_2$

$\therefore e^{-rT} E^Q [(K-d) I_{\{S_T \ge d, P_T \ge K-d\}}] = e^{-rT} (K-d) Q(-\dfrac{\Delta W_{sT}^Q}{\sqrt{T}} \le f_2, -\dfrac{\Delta W_{pT}^Q}{\sqrt{T}} \le g_2)$

$= e^{-rT} (K-d) BN(f_2, g_2, \rho)$

The second part of the price formula.

2. $e^{-rT} E^Q [(P_T + S_T - K) I_{\{S_T < d, P_T + S_T \ge K\}}]$

This part is like an Asian option. We follow Gentle (1994) and Vorst (1992) to derive the pricing model.

Rewrite $P_T + S_T - K$

$$P_T + S_T - K = [F_1 \frac{P_T}{F_1} + F_2 \frac{S_T}{F_2}] - K$$

$$= \{[\frac{F_1}{F_1+F_2} P'_T + \frac{F_2}{F_1+F_2} S'_T] - \frac{K}{F_1+F_2}\}(F_1+F_2)$$

$$= \{[X_1 P'_T + X_2 S'_T] - K'\}(F_1+F_2)$$

where,

$$X_1 = \frac{F_1}{F_1+F_2},\ \ X_2 = \frac{F_2}{F_1+F_2},\ \ \sum X_i = 1$$

$$K' = \frac{K}{F_1+F_2}$$

$$P'_T = \frac{P_T}{F_1},\ \ S'_T = \frac{S_T}{F_2}$$

$$F_1 = P_0 e^{rT},\ \ F_2 = S_0 e^{rT}$$

$$\therefore P_T + S_T - K = \{[X_1 P'_T + X_2 S'_T] - K'\}(F_1+F_2)$$

Following Vorst (1992),

$$X_1 P'_T + X_2 S'_T \approx (P'^{X1}_T S'^{X2}_T) - E(P'^{X1}_T S'^{X2}_T) + E(X_1 P'_T + X_2 S'_T)$$

where, $E(X_1 P'_T + X_2 S'_T) = 1 \ \Theta\ F_1 = E(P_T)$ and $F_2 = E(S_T)$

$$\therefore P_T + S_T - K = \{(P'^{X1}_T S'^{X2}_T) - E(P'^{X1}_T S'^{X2}_T) + E(X_1 P'_T + X_2 S'_T) - K'\}(F_1+F_2)$$

Let $K^* = K' + E(P'^{X1}_T S'^{X2}_T) - 1$

$$\therefore P_T + S_T - K = \{P'^{X1}_T S'^{X2}_T - K^*\}(F_1+F_2)$$

$$\Rightarrow e^{-rT} E^Q[(P_T + S_T - K) I_{\{S_T < d, P_T + S_T \geq K\}}]$$

$$\cong e^{-rT}(F_1+F_2) E^Q[(P'^{X1}_T S'^{X2}_T - K^*) I_{\{S_T < d, P'^{X1}_T S'^{X2}_T \geq K^*\}}]$$

Now, we will derive the mean and variance of $(P'_T{}^{X1} S'_T{}^{X2})$

$$P'_T{}^{X1} S'_T{}^{X2} = (\frac{P_T}{P_0 e^{rT}})^{X1}(\frac{S_T}{S_0 e^{rT}})^{X2} = (\frac{P_T}{P_0})^{X1}(\frac{S_T}{S_0})^{X2} e^{-rT(X1+X2)} = (\frac{P_T}{P_0})^{X1}(\frac{S_T}{S_0})^{X2} e^{-rT}$$

$$\ln(P'_T{}^{X1} S'_T{}^{X2}) = X_1 \ln(\frac{P_T}{P_0}) + X_2 \ln(\frac{S_T}{S_0}) - rT$$

Mean of $\ln(P'_T{}^{X1} S'_T{}^{X2})$

$$P_T = P_0 \exp[(r - \sigma_p^2/2)T + \sigma_p \Delta W_{pT}^Q \quad \text{and} \quad S_T = S_0 \exp[(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q]]$$

$$\therefore \alpha = E[\ln(P_T^{'X1} S_T^{'X2})] = X_1 E[\ln(\frac{P_T}{P_0})] + X_2 E[\ln(\frac{S_T}{S_0})] - rT$$

$$= X_1(r - \sigma_p^2/2)T + X_2(r - \sigma_S^2/2)T - rT = -(X_1 \sigma_p^2/2 + X_2 \sigma_S^2/2)T$$

Variance of $\ln(P'_T{}^{X1} S'_T{}^{X2})$

$$v^2 = Var[\ln(P'_T{}^{X1} S'_T{}^{X2})] = Var[X_1 \ln(\frac{P_T}{P_0}) + X_2 \ln(\frac{S_T}{S_0}) - rT]$$

$$= X_1{}^2 Var[\ln(\frac{P_T}{P_0})] + X_2{}^2 Var[\ln(\frac{S_T}{S_0})] + 2X_1 X_2 Cov(\ln(\frac{P_T}{P_0}), \ln(\frac{S_T}{S_0})]$$

$$= X_1{}^2 \sigma_P^2 + X_2{}^2 \sigma_S^2 + 2X_1 X_2 \rho \sigma_P \sigma_S$$

$\therefore$ Mean of $P'_T{}^{X1} S'_T{}^{X2}$, $E(P'_T{}^{X1} S'_T{}^{X2}) = e^{\alpha + v^2/2T}$

Also, we find

$$Cov(\ln[P'_T{}^{X1} S'_T{}^{X2}], \ln\frac{S_T}{S_0}) = Cov(X_1 \ln(\frac{P_T}{P_0}) + X_2 \ln(\frac{S_T}{S_0}) - rT, \ln\frac{S_T}{S_0})$$

$$= Cov(X_1 \ln(\frac{P_T}{P_0}), \ln\frac{S_T}{S_0}) + Cov(X_2 \ln(\frac{S_T}{S_0}), \ln\frac{S_T}{S_0})$$

$$= X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2$$

Now, back to $e^{-rT}(F_1 + F_2)E^Q[(P'_T{}^{X1} S'_T{}^{X2} - K^*) I_{\{S_T < d, P_T^{'X1} S_T^{'X2} \geq K^*\}}]$

By Girsanov Theorem, we view $P'_T{}^{X1} S'_T{}^{X2}$ as the first asset, and S as the second asset, we then have

$$P'_T{}^{X1} S'_T{}^{X2} = P'_0{}^{X1} S'_0{}^{X2} \exp[(r - v^2/2)T + v\Delta W_{vT}^Q] = P'_0{}^{X1} S'_0{}^{X2} \exp[(r + v^2/2)T + v\Delta W_{vT}^R]$$

$$S_T = S_0 \exp[(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q]$$

$$= S_0 \exp[(r - \sigma_s^2/2)T + (X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2)T + \sigma_s \Delta W_{sT}^R]$$

a. $e^{-rT}(F_1 + F_2)E^Q[(P'_T{}^{X1} S'_T{}^{X2}) I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*\}}]$

$$e^{-rT}(F_1 + F_2)E^Q[(P'_T{}^{X1} S'_T{}^{X2}) I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*\}}]$$

$$= (F_1 + F_2)(P'_0{}^{X1} S'_0{}^{X2}) P_r^R (P'_T{}^{X1} S'_T{}^{X2} \geq K^*, S_T < d)$$

$$= (F_1 + F_2) e^{-rt} E(P'_T{}^{X1} S'_T{}^{X2}) P_r^R (P'_T{}^{X1} S'_T{}^{X2} \geq K^*, S_T < d)$$

$$= (F_1 + F_2) e^{-rt} e^{\alpha + v^2/2T} P_r^R (P'_T{}^{X1} S'_T{}^{X2} \geq K^*, S_T < d)$$

Θ $P'_0{}^{X1} S'_0{}^{X2} = e^{-rt} E(P'_T{}^{X1} S'_T{}^{X2})$ given risk-neutral,

$$P'_T{}^{X1} S'_T{}^{X2} \geq K^*$$

$$\Rightarrow P'_T{}^{X1} S'_T{}^{X2} \exp[(r + v^2/2)T + v\Delta W_{vT}^R] \geq K^*$$

$$\Rightarrow \exp[(r + v^2/2)T + v\Delta W_{vT}^R] \geq \frac{K^*}{P'_0{}^{X1} S'_0{}^{X2}}$$

$$\Rightarrow -\frac{\Delta W_{vT}^R}{\sqrt{T}} \leq \frac{\ln(P'_0{}^{X1} S'_0{}^{X2}/K^*) + (r + v^2/2)T}{v\sqrt{T}} = f_3$$

$$f_3 = \frac{\ln(P'_0{}^{X1} S'_0{}^{X2}/K^*) + (r + v^2/2)T}{v\sqrt{T}} = \frac{\ln(e^{-rt} E(P'_T{}^{X1} S'_T{}^{X2})/K^*) + (r + v^2/2)T}{v\sqrt{T}}$$

$$= \frac{\ln E(P'_T{}^{X1} S'_T{}^{X2}) - \ln(K^*) - rT + (r + v^2/2)T}{v\sqrt{T}}$$

$$= \frac{\ln e^{\alpha + v^2/2T} - \ln(K^*) + v^2/2T}{v\sqrt{T}}$$

$$= \frac{\alpha - \ln(K^*) + v^2 T}{v\sqrt{T}}$$

$S_T < d$

➔ $S_0 \exp[(r - \sigma_s^2/2)T + (X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2)T + \sigma_s \Delta W_{sT}^R < d$

➔ $(r - \sigma_s^2/2)T + (X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2)T + \sigma_s \Delta W_{sT}^R < \ln(d/S_0)$

➔ $\frac{\Delta W_{sT}^R}{\sqrt{T}} < -\frac{\ln(S_0/d) + (r - \sigma_s^2/2 + X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2)T}{\sigma_s \sqrt{T}} = -g_3$

$\therefore e^{-rT}(F_1 + F_2)E^Q[(P'_T{}^{X1} S'_T{}^{X2})I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*\}}]$

$$= (F_1 + F_2)e^{-rt} e^{\alpha + v^2/2T} P_r^R (P'_T{}^{X1} S'_T{}^{X2} \geq K^*, S_T < d)$$

$$= (F_1 + F_2)e^{-rt} e^{\alpha + v^2/2T} BN(f_{3,} - g_3, \rho^*) \quad \text{where} \quad \rho^* = X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2$$

b. $e^{-rT}(F_1 + F_2)E^Q[(K^*)I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*\}}]$

$e^{-rT}(F_1 + F_2)E^Q[(K^*)I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*\}}]$

$$= e^{-rT}(F_1 + F_2)(K^*)Q(S_T < d, P'_T{}^{X1} S'_T{}^{X2} \geq K^*)$$

$P'_T{}^{X1} S'_T{}^{X2} \geq K^*$

➔ $P'_0{}^{X1} S'_0{}^{X2} \exp[(r - v^2/2)T + v\Delta W_{vT}^R] \geq K^*$

$\Rightarrow$ $\exp[(r - v^2/2)T + v\Delta W_{vT}^R] \geq \dfrac{K^*}{P'_0{}^{X1} S'_0{}^{X2}}$

$\Rightarrow$ $-\dfrac{\Delta W_{vT}^R}{\sqrt{T}} \leq \dfrac{\ln(P'_0{}^{X1} S'_0{}^{X2}/K^*) + (r - v^2/2)T}{v\sqrt{T}} = f_4$

$$f_4 = \frac{\ln(P'_0{}^{X1} S'_0{}^{X2}/K^*) + (r - v^2/2)T}{v\sqrt{T}} = \frac{\ln(e^{-rt} E(P'_T{}^{X1} S'_T{}^{X2})/K^*) + (r - v^2/2)T}{v\sqrt{T}}$$

$$= \frac{\ln E(P'_T{}^{X1} S'_T{}^{X2}) - \ln(K^*) - rT + (r - v^2/2)T}{v\sqrt{T}}$$

$$= \frac{\ln e^{\alpha + v^2/2T} - \ln(K^*) - v^2/2T}{v\sqrt{T}}$$

$$= \frac{\alpha - \ln(K^*)}{v\sqrt{T}}$$

$S_T < d$

$\Rightarrow$ $S_0 \exp[(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q] < d/S_0$

$\Rightarrow$ $(r - \sigma_s^2/2)T + \sigma_s \Delta W_{sT}^Q] < \ln(d/S_0)$

$\Rightarrow$ $\dfrac{\Delta W_{sT}^Q}{\sqrt{T}} < -\dfrac{\ln(S_0/d) + (r - \sigma_s^2/2)T}{\sigma_s\sqrt{T}} = -g_2$

$$\therefore e^{-rT}(F_1 + F_2)E^Q[(K*)I_{\{S_T < d, P_T'^{X1} S_T'^{X2} \geq K*\}}]$$

$$= e^{-rT}(F_1 + F_2)(K^*)Q(S_T < d, P_T'^{X1} S_T'^{X2} \geq K^*)$$

$$= e^{-rT}(F_1 + F_2)(K^*)BN(f_4, -g_2, \rho)$$

So, overall, the price of the purchase discount option is as following.

$$C_T = \max[P_T - K + d - \max(d - S_T, 0), 0]$$

$$C = e^{-rT} E^Q[(P_T - K + d) I_{\{S_T \geq d, P_T \geq K-d\}}] + e^{-rT} E^Q[(P_T - K + S_T) I_{\{S_T < d, P_T + S_T \geq K\}}]$$

$$= P_0 BN(f_1, g_1, \rho) - \ e^{-rT}(K - d) BN(f_2, g_2, \rho)$$

$$+ (F_1 + F_2) e^{-rt} e^{\alpha + v^2/2T} BN(f_{3,} - g_3, \rho^*) - \ e^{-rT}(F_1 + F_2)(K^*) BN(f_4, -g_2, \rho)$$

Where,

$$f_1 = \frac{\ln(P_0 / K - d) + (r + \sigma_p^2 / 2)T}{\sigma_p \sqrt{T}}, \quad g_1 = \frac{\ln(S_0 / d) + (r - \sigma_s^2 / 2 + \rho \sigma_s \sigma_p)T}{\sigma_s \sqrt{T}}$$

$$f_2 = \frac{\ln(\frac{P_0}{K - d}) + (r - \sigma_p^2 / 2)T}{\sigma_p \sqrt{T}}, \quad g_2 = \frac{\ln(S_0 / d) + (r - \sigma_s^2 / 2)T}{\sigma_s \sqrt{T}}$$

$$f_3 = \frac{\alpha - \ln(K^*) + v^2 T}{v\sqrt{T}}, \quad -g_3 = -\frac{\ln(S_0 / d) + (r - \sigma_s^2 / 2 + X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2)T}{\sigma_s \sqrt{T}}$$

$$f_4 = \frac{\alpha - \ln(K^*)}{v\sqrt{T}}$$

$$F_1 = P_0 e^{rT}, \quad F_2 = S_0 e^{rT}, \quad P'_T = \frac{P_T}{F_1}, \quad S'_T = \frac{S_T}{F_2}$$

$$X_1 = \frac{F_1}{F_1 + F_2}, \quad X_2 = \frac{F_2}{F_1 + F_2}, \quad \sum X_i = 1$$

$$K' = \frac{K}{F_1 + F_2}, \quad K^* = K' + E(P'_T{}^{X1} S'_T{}^{X2}) - 1$$

$$E(P'_T{}^{X1} S'_T{}^{X2}) = e^{\alpha + v^2/2T}$$

$$\alpha = -(X_1 \sigma_p^2 / 2 + X_2 \sigma_S^2 / 2)T$$

$$\rho^* = X_1 \rho \sigma_s \sigma_p + X_2 \sigma_S^2, \quad v^2 = X_1{}^2 \sigma_P^2 + X_2{}^2 \sigma_S^2 + 2 X_1 X_2 \rho \sigma_P \sigma_S$$

IV. CONCLUSIONS AND SUGGESTIONS

We design a house purchase discount options for developers to promote their projects. This new approach is a linkage between stock and house price. These options are found to be valuable to homebuyers. Therefore, they are helpful to developers to sell their house units and push up the developer's stock price. These options can be designed as European-style or American-style options. They are suitable to promote house sale in a pre-sale system. We are able to derive a closed-form pricing formula for these purchase discount options. This kind of discount options can have wide application to sell products with a link to an index or a stock's performance.

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